

Pairwise Almost Lindelöf Bitopological Spaces II

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ABSTRACT

In this paper we continuation to study the pairwise almost Lindelöf subspaces and subsets, investigate some of their characterizations and obtain some new results.

1. INTRODUCTION

The study of bitopological spaces was first initiated by J. C. Kelly [5] in 1963 and thereafter a large number of papers have been done in order to generalize the topological concepts to bitopological setting. In literature there are several generalizations of the notion of Lindelöf spaces and these are studied separately for different reasons and purposes. In 1984, Willard and Dissanayake [10] introduced and studied the notion of almost Lindelöf spaces and then in 1996, Commaroto and Santoro [2] studied and gave further new results about these spaces.

In our earlier paper [7], we have introduced and defined the notion of almost Lindelöf spaces in bitopological spaces, which we call pairwise almost Lindelöf spaces and investigate some of their properties. Further we also studied the pairwise almost Lindelöf subspaces and subsets and also investigated some of their further properties. This purpose of the present paper is to continue the study of these spaces and give more results concerning pairwise almost Lindelöf spaces, its subspaces as well as subsets.

2. PRELIMINARIES

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always meant as topological spaces and bitopological spaces, respectively unless explicitly stated. By i -open set, we shall means the open set with respect to topology τ_i in X . We always use (i, j) - to denote the certain properties with respect to topology τ_i and τ_j respectively, where $i, j \in \{1, 2\}$ and $i \neq j$. In this paper, every result in terms of (i, j) - will have pairwise as a corollary.

By i -int (A) and i -cl (A) we shall mean the interior and the closure of a subset A of X with respect to topology τ_i , respectively. We denote by int (A) and cl (A) for the interior and closure of a subset A of X with respect to topology τ_i for each $i = 1, 2$, respectively. The i -open cover of X , means that the cover of X by i -open sets in X ; similar for the (i, j) -regular open cover of X etc.

Definition 1 (see [6, 8]). A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j) -regular open (resp. (i, j) -regular closed) if $i\text{-int}(j\text{-cl}(S)) = S$ (resp. $i\text{-cl}(j\text{-int}(S)) = S$), where $i, j \in \{1, 2\}$ and $i \neq j$. S is said to be pairwise regular open (resp. pairwise regular closed) if it is both (i, j) -regular open and (j, i) -regular open (resp. (i, j) -regular closed and (j, i) -regular closed).

The topology generated by the (i, j) -regular open subsets of (X, τ_1, τ_2) is denoted by $\tau_{(i,j)}^s$ and it is called (i, j) -semiregularization of X . The topologies is pairwise semiregularization of X if one of the topologies is (i, j) -semiregularization of X and another topology is (j, i) -semiregularization of X . If $\tau_i \equiv \tau_{(i,j)}^s$ then X is said to be (i, j) -semiregular. Thus the space (X, τ_1, τ_2) is said pairwise semiregular if it is both (i, j) -semiregular and (j, i) -semiregular, that is, whenever $\tau_i \equiv \tau_{(i,j)}^s$, for each $i = 1, 2$ and $i \neq j$. In other words, (X, τ_1, τ_2) is (i, j) -semiregular if the family of (i, j) -regular open sets form a base for the topology τ_i . It is very clear that $\tau_{(i,j)}^s \subseteq \tau_i$, but it is not necessary that $\tau_i \subseteq \tau_{(i,j)}^s$. Now we recall the following several definitions that will be apply later to our results.

Definition 2 (see [1]). A bitopological space (X, τ_1, τ_2) is said to be i -Lindelöf if the topological space (X, τ_i) is Lindelöf. X is said to be Lindelöf if it is i -Lindelöf for each $i = 1, 2$. In other words, (X, τ_1, τ_2) is said to be Lindelöf if the topological space (X, τ_1) and (X, τ_2) are both Lindelöf.

Definition 3 (see [11]). A bitopological space X is said to be (i, j) -nearly Lindelöf if for every i -open cover $\{U_\alpha : \alpha \in \Delta\}$ of X , there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that, $X = \bigcup_{n \in \mathbb{N}} i\text{-int}(j\text{-cl}(U_{\alpha_n}))$ or equivalently, every (i, j) -regular open cover of X has a countable subcover. X is said to be pairwise nearly Lindelöf if it is both (i, j) -nearly Lindelöf and (j, i) -nearly Lindelöf.

Definition 4 (see [6, 9]). A bitopological space X is said to be (i, j) -almost regular if for each $x \in X$ and for each (i, j) -regular open set V of X containing x , there is an i -open set U such that $x \in U \subseteq j\text{-cl}(U) \subseteq V$. X is said to be pairwise almost regular if it is both (i, j) -almost regular and (j, i) -almost regular.

Definition 5 (see [5, 6]). A bitopological space (X, τ_1, τ_2) is said to be (i, j) -regular if for each point $x \in X$ and for each i -open set V containing x , there exists an i -open set U such that $x \in U \subseteq j\text{-cl}(U) \subseteq V$. X is said to be pairwise regular if it is both (i, j) -regular and (j, i) -regular.

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Definition 6. Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be:

- (i) open if F is both 1-open and 2-open in X , or equivalently, $F \in U$ for $U \subseteq (\tau_i \cap \tau_2)$ in X .
- (ii) closed if F is both 1-closed and 2-closed in X , or equivalently, $F \in V$ for $V = X/U$ and $U \subseteq (\tau_i \cap \tau_2)$ in X .
- (iii) clopen if F is both 1-clopen and 2-clopen in X .

Definition 7 (see [6, 8]). A bitopological space X is said to be (i, j) -semiregular if for each $x \in X$ and for each i -open set V of X containing x , there is an i -open set U such that $x \in U \subseteq i - \text{int}(j - \text{cl}(U)) \subseteq V$. X is said to be pairwise semi regular if it is both (i, j) -semi regular and (j, i) -semi regular.

Datta in [3] introduced the concept of pairwise extremally disconnectedness in bitopology as follows:

Definition 8. A bitopological space (X, τ_1, τ_2) is said to be (i, j) -extremally disconnected if the i -closure of every j -open set is j -open. X is said to be pairwise extremally disconnected if it is both (i, j) -extremally disconnected and (j, i) -extremally disconnected.

3. PAIRWISE ALMOST LINDELÖF SPACES

We begin with defining a pairwise almost Lindelöf spaces as stated in [7].

Definition 9. A bitopological space X is said to be (i, j) -almost Lindelöf if for every i -open cover $\{U_\alpha : \alpha \in \Delta\}$ of X , there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbb{N}} j - \text{cl}(U_{\alpha_n})$. X is said to be pairwise almost Lindelöf if it is both (i, j) -almost Lindelöf and (j, i) -almost Lindelöf.

The following propositions generalize the results in an earlier paper, see [7].

Proposition 1. A bitopological space X is (i, j) -almost Lindelöf if and only if every family $\{C_{\alpha_n} : n \in \Delta\}$ of i -closed subsets of X such that $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$ admits a countable subfamily such that $\bigcap_{n \in \mathbb{N}} j - \text{int}(C_{\alpha_n}) = \emptyset$.

Proof. If $\{C_\alpha : \alpha \in \Delta\}$ is a family of i -closed subsets of X such that $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$, then $X = X \setminus \bigcap_{\alpha \in \Delta} C_\alpha = \bigcup_{\alpha \in \Delta} (X \setminus C_\alpha)$, i.e., the family $\{X \setminus C_\alpha : \alpha \in \Delta\}$ is an i -open cover of X . Since X is (i, j) -almost Lindelöf, there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n \in \mathbb{N}} j - \text{cl}(X \setminus C_{\alpha_n})$. So $X \setminus \bigcup_{n \in \mathbb{N}} j - \text{cl}(X \setminus C_{\alpha_n}) = \emptyset$, i.e., $X \setminus \bigcup_{n \in \mathbb{N}} j - \text{cl}(X \setminus C_{\alpha_n}) = \emptyset$ or $\bigcap_{n \in \mathbb{N}} j - \text{int}(C_{\alpha_n}) = \emptyset$

Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an i -open cover of X . Then $X = \bigcup U_\alpha$ and $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of i -closed subsets of X . Hence $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \emptyset$, i.e., $\bigcap_{\alpha \in \Delta} (X \setminus U_\alpha) = \emptyset$. By hypothesis, there exists a countable $\alpha \in \Delta$ subfamily $\{X \setminus U_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) = \emptyset$. So $X = X \setminus \bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) = \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$. Therefore X is (i, j) -almost Lindelöf. \square

Corollary 1. *A bitopological space X is pairwise almost Lindelöf if and only if every family $\{C_\alpha : \alpha \in \Delta\}$ of closed subsets of X such that $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$ admits a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) = \emptyset$.*

Proof. It is obvious by the definition. \square

Proposition 2. *A bitopological space X is (i, j) -almost Lindelöf if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ by i -closed subsets of X , there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) = \emptyset$, the intersection $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$.*

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of i -closed subsets of X for which there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) = \emptyset$. Suppose that $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$. Hence $X = X \setminus \bigcap_{\alpha \in \Delta} C_\alpha = \bigcup_{\alpha \in \Delta} (X \setminus C_\alpha)$. Thus $\{X \setminus C_\alpha : \alpha \in \Delta\}$ forms an i -open cover for X . Since X is (i, j) -almost Lindelöf, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbb{N}} j\text{-cl}(X \setminus C_{\alpha_n})$. Hence $X \setminus \bigcup_{n \in \mathbb{N}} j\text{-cl}(X \setminus C_{\alpha_n}) = \emptyset$, i.e. $X \setminus \bigcup_{n \in \mathbb{N}} (X \setminus j\text{-int}(C_{\alpha_n})) = \emptyset$ or $\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) = \emptyset$ which is a contradiction. Conversely, suppose that X is not (i, j) -almost Lindelöf. Then there exists an i -open cover $\{U_\alpha : \alpha \in \Delta\}$ of X with no countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$. Hence $X \neq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$ for any countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$. It follows that $X \setminus \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n}) \neq \emptyset$, i.e., $\bigcap_{n \in \mathbb{N}} (X \setminus j\text{-cl}(U_{\alpha_n})) \neq \emptyset$ or $\bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) \neq \emptyset$. Thus $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of i -closed subsets of X that satisfies $\bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) \neq \emptyset$ for a countable subfamily $\{X \setminus U_{\alpha_n} : n \in \mathbb{N}\}$. Now by hypothesis, the intersection $\bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) \neq \emptyset$, and it follows that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \neq \emptyset$, i.e., $X \neq \bigcup_{\alpha \in \Delta} U_\alpha$, this is a contradiction with the fact that $\{U_\alpha : \alpha \in \Delta\}$ is an i -open cover of X . Therefore X is (i, j) -almost Lindelöf. \square

Corollary 2. *A bitopological space X is pairwise almost Lindelöf if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ by closed subsets of X , there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha \in \Delta} C_\alpha \neq \emptyset$.*

The following results can be obtained from an earlier paper [7].

Proposition 3. *Let (X, τ_1, τ_2) be a bitopological space. If the following conditions are satisfied:*

- (i) X is (i, j) -almost Lindelöf,
 - (ii) every (i, j) -regular open cover $\{U_\alpha : \alpha \in \Delta\}$ of X admits a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n \in \mathbb{N}} \text{cl}(U_{\alpha_n})$;
 - (iii) every family $\{C_\alpha : \alpha \in \Delta\}$ of (i, j) -regular closed subsets of X such that $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$ admits a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) \neq \emptyset$;
 - (iv) every family $\{C_\alpha : \alpha \in \Delta\}$ of (i, j) -regular closed subsets of X with countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha \in \Delta} C_\alpha \neq \emptyset$;
- then we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and if X is (i, j) -semiregular, then we have $(ii) \Rightarrow (i)$.

Corollary 3. *Let (X, τ_1, τ_2) be a bitopological space. For the following conditions*

- (i) X is pairwise almost Lindelöf
- (ii) every pairwise regular open cover $\{U_\alpha : \alpha \in \Delta\}$ of X admits a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n \in \mathbb{N}} \text{cl}(U_{\alpha_n})$;
- (iii) every family $\{C_\alpha : \alpha \in \Delta\}$ of pairwise regular closed subsets of X such that $\bigcap_{\alpha \in \Delta} C_\alpha \neq \emptyset$ admits a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) \neq \emptyset$;
- (iv) every family $\{C_\alpha : \alpha \in \Delta\}$ of pairwise regular closed subsets of X with countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha \in \Delta} C_\alpha \neq \emptyset$;

we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and if X is pairwise semiregular, then $(ii) \Rightarrow (i)$. Obviously, if a space is (i, j) -nearly Lindelöf, then it is (i, j) -almost Lindelöf since

$$\bigcup_{n \in \mathbb{N}} i - \text{int}(j - \text{cl}(U_{\alpha_n})) \subseteq \bigcup_{n \in \mathbb{N}} j - \text{cl}(U_{\alpha_n}).$$

Question 1. Does (i, j) -almost Lindelöf property imply (i, j) -nearly Lindelöf property?

The author expect that the answer of this question is negative as in the single topology, see [4].

Proposition 4. *An (i, j) -almost regular space is (i, j) -almost Lindeöf if and only if it is (i, j) -nearly Lindelöf.*

Corollary 4. *A pairwise almost regular space is pairwise almost Lindelöf if and only if it is pairwise nearly Lindelöf.*

The above result shows that in pairwise almost regular spaces, pairwise almost Lindelöf property and pairwise nearly Lindelöf property are equivalent. Other extra results are the following.

Lemma 1. *If (X, τ_1, τ_2) is (j, i) -extremally disconnected, then it is (i, j) -almost regular.*

Proof. Let $x \in X$ and let V be an (i, j) -regular open subset of X containing x . Hence V is also an i -open subset of X containing x . Since X is (j, i) -extremally disconnected, $j\text{-cl}(V)$ is i -open and so $j\text{-cl}(V) = i\text{-int}(j\text{-cl}(V)) = V$. Thus V is an (i, j) -regular open subset of X such that $x \in V \subseteq j\text{-cl}(V) \subseteq V$. So X is (i, j) -almost regular by definition 4.

Corollary 5. *If (X, τ_1, τ_2) is pairwise extremally disconnected, then it is pairwise almost regular.*

On using Proposition 4 and Lemma 1, we conclude the following proposition.

Proposition 5. *A (j, i) -extremally disconnected space is (i, j) -almost Lindelöf if and only if it is (i, j) -nearly Lindelöf.*

Corollary 6. *A pairwise extremally disconnected space is pairwise almost Lindelöf if and only if it is pairwise nearly Lindelöf.*

The converse of Proposition 5 is not true by the following counter-example.

Example 1. Consider $(\mathbb{R}, \tau_1, \tau_2)$ where \mathbb{R} is the real line, τ_1 is usual topology and τ_2 is a right hand topology, i.e., a topology generated by $\{(x, \infty) : x \in \mathbb{R}\}$. Observe that \mathbb{R} is $(2, 1)$ -almost regular since the only $(2, 1)$ -regular open subsets of \mathbb{R} are sets of the form $(x, \infty) : x \in \mathbb{R}$. But it is not $(1, 2)$ -extremally disconnected since $(3, \infty)$ is 2 -open set in \mathbb{R} and $1\text{-cl}((3, \infty)) = [3, \infty)$ is not 2 -open set.

Lemma 2 (see [11]). *An (i, j) -semiregular space X is an (i, j) -nearly Lindelöf if and only if it is i -Lindelöf.*

The direct consequence of Proposition 5 and Lemma 2 yield the following corollary.

Corollary 7. *A (j, i) -extremally disconnected and (i, j) -semiregular space is (i, j) -almost Lindelöf if and only if it is i -Lindelöf.*

Corollary 8. *A pairwise extremally disconnected and pairwise semiregular space is pairwise almost Lindelöf if and only if it is Lindelöf.*

The proof of the following proposition can be obtained in [7]. However by using Proposition 4 and Lemma 2, one can also obtain a direct proof of the following result since every (i, j) -regular space is (i, j) -almost regular and also (i, j) -semiregular.

Proposition 6. *An (i, j) -regular space is (i, j) -almost Lindelöf if and only if it is i -Lindelöf.*

Corollary 9. *A pairwise regular space is pairwise almost Lindelöf if and only if it is Lindelöf.*

Pairwise Almost Lindelöf Subspaces and Subsets

Recalling from [7], a subset S of a bitopological space X is said to be (i, j) -almost Lindelöf (resp. pairwise almost Lindelöf) if S is (i, j) -almost Lindelöf (resp. pairwise almost Lindelöf) as a subspace of X , i.e., S is (i, j) -almost Lindelöf (resp. pairwise almost Lindelöf) with respect to the induced bitopology from the bitopology of X and the following is a definition of (i, j) -almost Lindelöf relative to a bitopological space, see [7].

Definition 10. *A subset S of a bitopological space X is said to be (i, j) -almost Lindelöf relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i -open sets of X such that $S \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$. S is said to be pairwise almost Lindelöf relative to X if S is both (i, j) -almost Lindelöf relative to X and (j, i) -almost Lindelöf relative to X .*

Now we prove the following results.

Proposition 7. *Let S be a subset of a bitopological space X . Then S is (i, j) -almost Lindelöf relative to X if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ of i -closed subsets of X such that*

$$\left(\bigcap_{\alpha \in \Delta} C_\alpha \right) \cap S = \emptyset, \text{ there exists a countable subfamily } \{C_{\alpha_n} : n \in \mathbb{N}\} \text{ such that}$$

$$\left(\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) \right) \cap S = \emptyset.$$

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of i -closed subsets of X such that $\left(\bigcap_{\alpha \in \Delta} C_\alpha \right) \cap S = \emptyset$. Then $S \subseteq X \setminus \left(\bigcap_{\alpha \in \Delta} C_\alpha \right) = \bigcup_{\alpha \in \Delta} (X \setminus C_\alpha)$, so $\{X \setminus C_\alpha : \alpha \in \Delta\}$ forms a family of i -open subsets

of X covering S . By hypothesis, there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(X \setminus C_{\alpha_n})$. Hence

$$\left(X \setminus \bigcup_{n \in \mathbb{N}} j\text{-cl}(X \setminus C_{\alpha_n}) \right) \cap S = \emptyset, \text{ i.e., } \left(X \setminus \bigcup_{n \in \mathbb{N}} (X \setminus j\text{-int}(C_{\alpha_n})) \right) \cap S = \emptyset$$

Thus $\left(\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) \right) \cap S = \emptyset$. Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be a family of i -open sets in X such that $S \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Then $\left(X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \right) \cap S = \emptyset$, i.e., $\left(\bigcap_{\alpha \in \Delta} X \setminus U_\alpha \right) \cap S = \emptyset$. Since $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of i -closed subsets of X , by hypothesis there exists a countable subfamily $\{X \setminus U_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$\left(\bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) \right) \cap S = \emptyset, \text{ i.e., } \left(\bigcap_{n \in \mathbb{N}} X \setminus j\text{-cl}(U_{\alpha_n}) \right) \cap S = \emptyset.$$

Therefore $\left(X \setminus \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n}) \right) \cap S = \emptyset$, i.e., $S \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$. This completes the proof. □

Corollary 10. *Let S be a subset of a bitopological space X . Then S is pairwise almost Lindelöf relative to X if and only if for every family $\{C_\alpha : \alpha \in \Delta\}$ of closed subsets of X such*

that $\left(\bigcap_{\alpha \in \Delta} C_\alpha \right) \cap S = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n}) \right) \cap S = \emptyset$.

Proposition 8. *Let X be a bitopological space and $S \subseteq X$. For the following conditions*

- (i) S is (i, j) -almost Lindelöf relative to X
- (ii) every family of (i, j) -regular open subsets $\{U_\alpha : \alpha \in \Delta\}$ of X that cover S admits a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$
- (iii) every family $\{C_\alpha : \alpha \in \Delta\}$ of (i, j) -regular closed subsets of X such that

$$\left(\bigcap_{\alpha \in \Delta} C_\alpha \right) \cap S = \emptyset$$

admits a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) \right) \cap S = \emptyset$;

- (iv) every family $\{C_\alpha : \alpha \in \Delta\}$ of (i, j) -regular closed subsets of X with countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n}) \right) \cap S = \emptyset$, the intersection $\left(\bigcap_{\alpha \in \Delta} C_\alpha \right) \cap S \neq \emptyset$; we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and if X is (i, j) -semiregular, then $(ii) \Rightarrow (i)$.

Proof. (i) \Rightarrow (ii): It is obvious from the definition since an (i, j) -regular open set is also i -open set.

(ii) \Leftrightarrow (iii). If $\{C_\alpha : \alpha \in \Delta\}$ is a family of (i, j) -regular closed subsets of X such that $\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) \cap S \neq \emptyset$, then $S \subseteq X \setminus \bigcap_{\alpha \in \Delta} C_\alpha = \bigcup_{\alpha \in \Delta} (X \setminus C_\alpha)$, i.e., the family $\{X \setminus C_\alpha : \alpha \in \Delta\}$ is an (i, j) -regular open subsets of X that cover S . By (ii), there exists a countable subfamily such that

$$S \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(X \setminus C_{\alpha_n}) = \bigcup_{n \in \mathbb{N}} X \setminus j\text{-int}(C_{\alpha_n}) = X \setminus \bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n})$$

So, $\left(\bigcap_{n \in \mathbb{N}} j\text{-int}(C_{\alpha_n})\right) \cap S = \emptyset$. Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be a family of (i, j) -regular open subsets of X that cover S . Then $S \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of (i, j) -regular closed subsets of X . Hence

$$\left(X \setminus \bigcup_{\alpha \in \Delta} U_\alpha\right) \cap S = \emptyset, \text{ i.e., } \left(\bigcap_{\alpha \in \Delta} (X \setminus U_\alpha)\right) \cap S = \emptyset$$

By (iii), there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$\left(\bigcap_{\alpha \in \Delta} j\text{-int}(X \setminus U_{\alpha_n})\right) \cap S = \emptyset$$

$$\text{So } S \subseteq X \setminus \bigcap_{n \in \mathbb{N}} j\text{-int}(X \setminus U_{\alpha_n}) = X \setminus \bigcap_{n \in \mathbb{N}} (X \setminus j\text{-cl}(U_{\alpha_n})) = \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$$

(iii) \Leftrightarrow (iv): Straight forward by taking complement.

(ii) \Rightarrow (i): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of i -open subsets of X that cover S . Since X is (i, j) -semiregular, we can assume that is (i, j) -regular open set for each. By (ii), there exists a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$. This completes the proof. □

Corollary 11. Let X be a bitopological space and $S \subseteq X$. For the following conditions

(i) S is pairwise almost Lindelöf relative to X

(ii) every family of pairwise regular open subsets $\{U_\alpha : \alpha \in \Delta\}$ of X that cover S admits a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that $S \subseteq \bigcup_{n \in \mathbb{N}} \text{cl}(U_{\alpha_n})$.

(iii) every family $\{C_\alpha : \alpha \in \Delta\}$ of pair wise regular closed subsets of X satisfying the condition

$\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) \cap S = \emptyset$ admits a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$\left(\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n})\right) \cap S = \emptyset$$

(iv) every family $\{C_\alpha : \alpha \in \Delta\}$ of pair wise regular closed subsets of X with countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n \in \mathbb{N}} \text{int}(C_{\alpha_n})\right) \cap S \neq \emptyset$, the intersection $\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) \cap S \neq \emptyset$;

we have that (i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) and if X is pairwise semiregular, then (ii) \Rightarrow (i).

The following results were proved in [7] and we restated here without proof.

Proposition 9. *Let X be a bitopological space and A be any subset of X . If A is (i, j) -almost Lindelöf, then it is (i, j) -almost Lindelöf relative to X .*

Corollary 12. *Let X be a bitopological space and A be any subset of X . If A is pairwise almost Lindelöf, then it is pairwise almost Lindelöf relative to X .*

Question 2. Is the converse of Proposition 9 above true?

The author expected that the answer is no however if $A \subseteq X$ is i -open, then the converse of Proposition 9 holds as follows.

Proposition 10. *Let X be a bitopological space and A an i -open subset of X . Then A is (i, j) -almost Lindelöf if and only if it is (i, j) -almost Lindelöf relative to X .*

Corollary 13. *Let X be a bitopological space and A an open subset of X . Then A is pairwise almost Lindelöf if and only if it is pairwise almost Lindelöf relative to X .*

The space X in Proposition 9, Proposition 10, Corollary 13 and Corollary 14 are correct in arbitrary bitopological spaces. In particular, if we consider X itself is an (i, j) -almost Lindelöf space, we have the following results.

Proposition 11. *Every (i, j) -regular closed and j -open subset of an (i, j) -almost Lindelöf and (i, j) -semiregular space X is (i, j) -almost Lindelöf relative to X .*

Corollary 14. *Every pairwise regular closed and open subset of a pairwise almost Lindelöf and pairwise semiregular space X is pairwise almost Lindelöf relative to X .*

Proposition 12. *An i -clopen and j -open subset of an (i, j) -almost Lindelöf space X is (i, j) -almost Lindelöf.*

Corollary 15. *A clopen subset of a pairwise almost Lindelöf space X is pairwise almost Lindelöf.*

Question 3. Is i -closed subset of an (i, j) -almost Lindelöf space X is (i, j) -almost Lindelöf?

Question 4. Is (i, j) -regular open subset of an (i, j) -almost Lindelöf space X is (i, j) -almost Lindelöf?

The author expect that the answers of both questions are negative. Observe that, the condition in Proposition 11 that a subset should be (i, j) -regular closed and j -open and in Proposition 12 that a subset should be i -clopen and j -open are necessary but not sufficient to be only i -open ([7], Example 2). In general, arbitrary subsets of (i, j) -almost Lindelöf spaces need not be (i, j) -almost Lindelöf relative to the spaces and so not (i, j) -almost Lindelöf by Proposition 9.

So we can say that in general, an (i, j) -almost Lindelöf property is not a hereditary property and therefore pairwise almost Lindelöf property is not so.

Definition 11. A bitopological space X is said to be hereditary (i, j) -almost Lindelöf if every subspace of X is (i, j) -almost Lindelöf. X is said to be hereditary pairwise almost Lindelöf if X is both hereditary (i, j) -almost Lindelöf and hereditary (j, i) -almost Lindelöf.

Now we give another new result as an extension to the results in [7].

Proposition 13. Let X be an (i, j) -semiregular bitopological space. Then X is i -open hereditary (i, j) -almost Lindelöf if and only if any $A \in \tau_{(i,j)}^s$ is (i, j) -almost Lindelöf.

Proof. Let X be an (i, j) semiregular and i -open hereditary (i, j) -almost Lindelöf space. Since $\tau_{(i,j)}^s \subseteq \tau_i$, it is obvious that any $A \in \tau_{(i,j)}^s$ implies $A \in \tau_i$ and hence A is (i, j) -almost Lindelöf. Conversely, let $B \subseteq X$ be an i -open subset of X . By Proposition 10 it is sufficient to prove that B is (i, j) -almost Lindelöf relative to X . Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a family by (i, j) -regular open subsets of X such that $B \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Thus by hypothesis, the set $A = \bigcup_{\alpha \in \Delta} U_\alpha \in \tau_{(i,j)}^s$, and follows that (i, j) -almost Lindelöf. Hence there exists a countable subfamily $\{U_{\alpha_n} : n \in \mathbb{N}\}$ of \mathcal{U} such that $A \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$ and therefore $B \subseteq \bigcup_{n \in \mathbb{N}} j\text{-cl}(U_{\alpha_n})$. Therefore B is (i, j) -almost Lindelöf relative to X since X is (i, j) -semiregular by Proposition 8. This completes the proof. □

Corollary 16. Let X be a pairwise semiregular bitopological space. Then X is open hereditary pairwise almost Lindelöf if and only if any A both belong to $\tau_{(i,j)}^s$ and $\tau_{(j,i)}^s$ is pairwise almost Lindelöf.

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